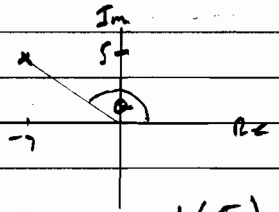
MEI FP2 COMPLEX EXERCISE 3I

1) $w^2 = -7 + 5j$

$\sqrt{(-7)^2 + 5^2} = \sqrt{74}$



$\theta = \pi - \tan^{-1}\left(\frac{5}{7}\right)$

$\theta = 2.52134$

$w^2 = \sqrt{74}\left(\cos 2.52134 + j \sin 2.52134\right)$

$w = (74)^{\frac{1}{4}}\left(\cos\left(\frac{2.52134}{2}\right) + j \sin\left(\frac{2.52134}{2}\right)\right)$

$\underline{w = 0.90 + 2.79j}$

Also

$w = (74)^{\frac{1}{4}}\left(\cos\left(\frac{2\pi + 2.52134}{2}\right) + j \sin\left(\frac{2\pi + 2.52134}{2}\right)\right)$

$\underline{w = -0.90 - 2.79j}$

2) $w^4 = -4$

$w^4 = 4\left(\cos \pi + j \sin \pi\right)$

$w = 4^{\frac{1}{4}}\left(\cos\left(\frac{2n\pi + \pi}{4}\right) + j \sin\left(\frac{2n\pi + \pi}{4}\right)\right)$

$w_1 = \sqrt{2}\left(\cos \frac{\pi}{4} + j \sin \frac{\pi}{4}\right)$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}}\right)$

$\underline{w_1 = 1 + j}$

$w_2 = \sqrt{2}\left(\cos \frac{3\pi}{4} + j \sin \frac{3\pi}{4}\right)$

$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + j \frac{1}{\sqrt{2}}\right)$

$\underline{w_2 = -1 + j}$

$w_3 = \sqrt{2}\left(\cos \frac{5\pi}{4} + j \sin \frac{5\pi}{4}\right)$

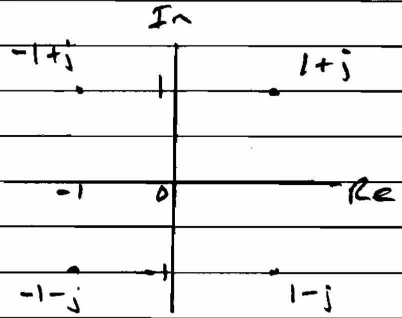
$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}}\right)$

$\underline{w_3 = -1 - j}$

$w_4 = \sqrt{2}\left(\cos \frac{7\pi}{4} + j \sin \frac{7\pi}{4}\right)$

$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - j \frac{1}{\sqrt{2}}\right)$

$\underline{w_4 = 1 - j}$



3) $w_1 = 2 + 3j$

$W = w_1^4 = (2 + 3j)^4$

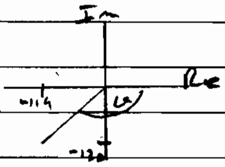
$= (4 + 12j - 9)^2$

$= (-5 + 12j)^2$

$= (25 - 120j - 144)$

$\underline{W = -119 - 120j}$

$w^4 = \sqrt{28561}\left(\cos \theta + j \sin \theta\right)$



$\theta = -\pi + \tan^{-1}\left(\frac{120}{119}\right)$

$\theta = -2.352$

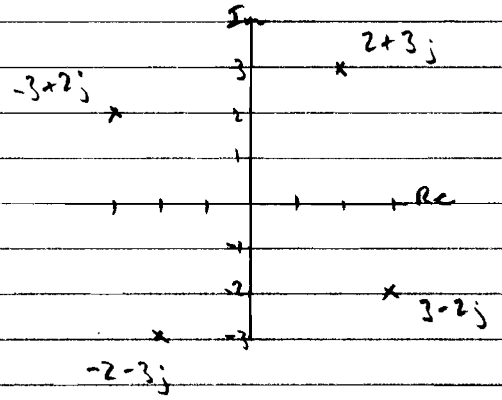
$w = 28561^{\frac{1}{8}}\left(\cos \frac{2n\pi - 2.352}{4} + j \sin \frac{2n\pi - 2.352}{4}\right)$

$w_1 = 3 - 2j$

$w_2 = 2 + 3j$

$w_3 = -3 + 2j$

$w_4 = -2 - 3j$



$w \approx -119 - 120j$ too far away to put on diagram

4) $(z - 3j)^5 = 32$

Find 5^{th} roots of 32

$32 = 32(\cos 0 + j \sin 0)$

$w = 32^{\frac{1}{5}}\left(\cos \frac{2n\pi}{5} + j \sin \frac{2n\pi}{5}\right)$

$w_1 = 2$

$w_2 = 2\left(\cos \frac{2\pi}{5} + j \sin \frac{2\pi}{5}\right)$

$w_2 = 0.618 + 1.902j$

$w_3 = 2\left(\cos \frac{4\pi}{5} + j \sin \frac{4\pi}{5}\right)$

$w_3 = -1.618 + 1.176j$

$w_4 = 2\left(\cos \frac{6\pi}{5} + j \sin \frac{6\pi}{5}\right)$

$w_4 = -1.618 - 1.176j$

$w_5 = 2\left(\cos \frac{8\pi}{5} + j \sin \frac{8\pi}{5}\right)$

$w_5 = 0.618 - 1.902j$

Solutions

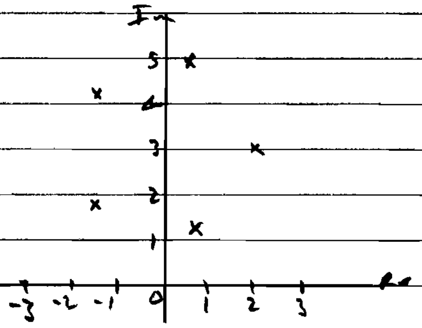
$z_1 = w_1 + 3j = 2 + 3j$

$z_2 = w_2 + 3j = 0.618 + 4.902j$

$z_3 = w_3 + 3j = -1.618 + 4.176j$

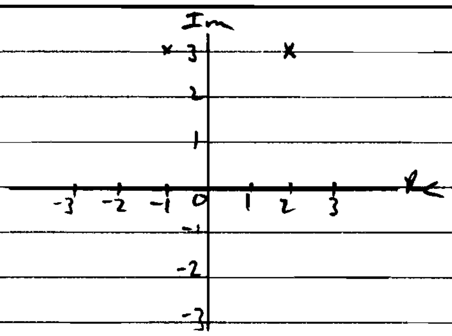
$z_4 = w_4 + 3j = -1.618 + 1.824j$

$z_5 = w_5 + 3j = 0.618 + 1.098j$



(Vertices of regular pentagon)

5)



$z = -1 + 3j + 3\left(\cos \frac{2k\pi}{7} + j \sin \frac{2k\pi}{7}\right)$

$(z + 1 - 3j)^7 = 3^7(\cos 0 + j \sin 0)$

$(z + 1 - 3j)^7 = 2187$

MEI FP2 COMPLEX EXERCISE 3I

9) $\alpha^4 = -64$

i) $\alpha^4 = 64(\cos\pi + j\sin\pi)$

$\alpha = 64^{\frac{1}{4}}\left(\cos\frac{2k\pi+\pi}{4} + j\sin\frac{2k\pi+\pi}{4}\right)$

for $k = 0, 1, 2, 3$

$\alpha_1 = \sqrt{8}\left(\cos\frac{\pi}{4} + j\sin\frac{\pi}{4}\right)$

$\alpha_1 = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$

$\underline{\alpha_1 = 2 + 2j}$

$\alpha_2 = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4}\right)$

$\alpha_2 = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}j\right)$

$\underline{\alpha_2 = -2 + 2j}$

$\alpha_3 = 2\sqrt{2}\left(\cos\frac{5\pi}{4} + j\sin\frac{5\pi}{4}\right)$

$\alpha_3 = 2\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}j\right)$

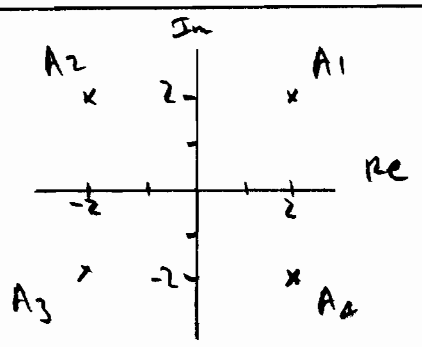
$\underline{\alpha_3 = -2 - 2j}$

$\alpha_4 = 2\sqrt{2}\left(\cos\frac{7\pi}{4} + j\sin\frac{7\pi}{4}\right)$

$\alpha_4 = 2\sqrt{2}\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$\underline{\alpha_4 = 2 - 2j}$

9 ii)

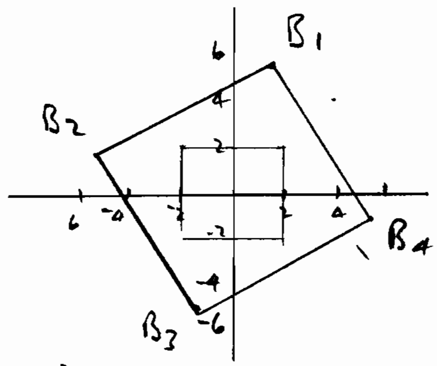


iii) $\beta = \sqrt{3} + j$

$|\beta| = \sqrt{\sqrt{3}^2 + 1^2} = 2$

$\arg\beta = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$

Multiplying by β enlarges about O by scale factor 2 and rotates by $\frac{\pi}{6}$ anti-clockwise



$(2+2j)(\sqrt{3}+j)$

$= 2\sqrt{3} + 2\sqrt{3}j + 2j - 2$

$= 1.46 + 5.46j \quad = B_1$

Small square has side 4 units

So large square has side $2 \times 4 = 8$ units

9 iv)

$\alpha_1\beta = (2+2j)(\sqrt{3}+j)$

$= \sqrt{8}e^{j\frac{\pi}{4}} \times 2e^{j\frac{\pi}{6}}$

$= 2\sqrt{8}e^{j\frac{5\pi}{12}}$

$(\alpha_1\beta)^4 = 2^4 \times 8^2 e^{j\frac{20}{12}\pi}$

$= 1024\left(\cos\frac{5\pi}{3} + j\sin\frac{5\pi}{3}\right)$

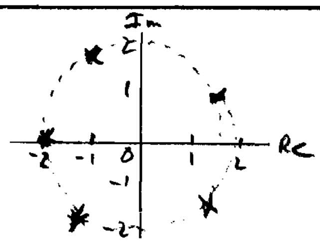
$= 1024\left(\frac{1}{2} - \frac{\sqrt{3}}{2}j\right)$

$= 512 - 512\sqrt{3}j$

MEI FP2 COMPLEX EXERCISE 3I

10 i)
$$e^{j\theta} = \cos\theta + j\sin\theta$$
$$e^{-j\theta} = \cos\theta - j\sin\theta$$
$$\frac{1}{1+e^{2j\theta}} = \frac{e^{-j\theta}}{e^{-j\theta}+e^{j\theta}}$$
$$= \frac{\cos\theta - j\sin\theta}{(\cos\theta - j\sin\theta)+(\cos\theta + j\sin\theta)}$$
$$= \frac{\cos\theta - j\sin\theta}{2\cos\theta}$$
$$= \frac{1}{2} - \frac{1}{2}j\tan\theta$$
$$= \frac{1}{2}\left(1 - j\tan\theta\right)$$

10 ii)
$$z^5 + 32 = 0$$
$$z^5 = -32$$
$$z^5 = 32(\cos\pi + j\sin\pi)$$
$$z = 32^{\frac{1}{5}}\left(\cos\frac{2k\pi+\pi}{5} + j\sin\frac{2k\pi+\pi}{5}\right)$$
$$z_1 = 2e^{j\frac{\pi}{5}}$$
$$z_2 = 2e^{j\frac{3\pi}{5}}$$
$$z_3 = 2e^{j\pi}$$
$$z_4 = 2e^{j\frac{7\pi}{5}} = 2e^{-\frac{3\pi}{5}}$$
$$z_5 = 2e^{j\frac{9\pi}{5}} = 2e^{-j\frac{\pi}{5}}$$



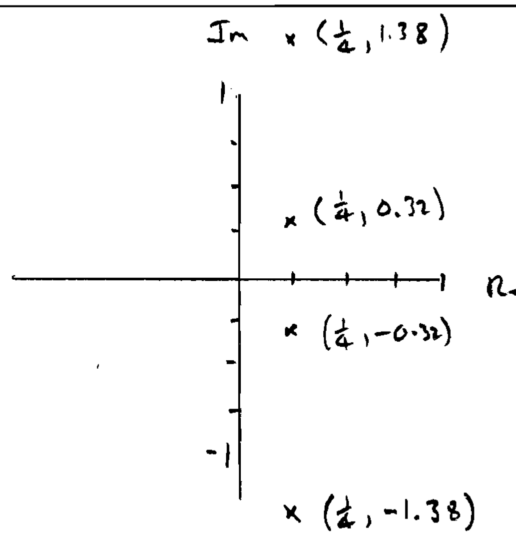
10 iii)
$$\left(\frac{1-2w}{w}\right)^5 + 32 = 0$$

Let $\frac{1-2w}{w} = z$
$$1 - 2w = zw$$
$$1 = zw + 2w$$
$$1 = w(z+2)$$
$$w = \frac{1}{z+2}$$
$$w = \frac{1}{2+2e^{j\alpha}}$$

where $\alpha = \pi, \pm\frac{\pi}{5}, \pm\frac{3\pi}{5}$
$$w = \frac{1}{2}\left(\frac{1}{1+e^{j\alpha}}\right) = \frac{1}{4}\left(1 - j\tan\frac{\alpha}{2}\right)$$

Possible values of $\frac{\alpha}{2}$

in interval $-\frac{1}{2}\pi < \frac{\alpha}{2} < \frac{\pi}{2}$
$$= \pm\frac{\pi}{10}, \pm\frac{3\pi}{10}$$



1) i) $e^{jk\theta} = \cos k\theta + j\sin k\theta$

$e^{-jk\theta} = \cos k\theta - j\sin k\theta$

$$\frac{1}{1-e^{j\theta}} = \frac{e^{-j\frac{\theta}{2}}}{e^{-j\frac{\theta}{2}} - e^{j\frac{\theta}{2}}}$$

$$= \frac{\cos\frac{\theta}{2} - j\sin\frac{\theta}{2}}{\left(\cos\frac{\theta}{2} - j\sin\frac{\theta}{2}\right) - \left(\cos\frac{\theta}{2} + j\sin\frac{\theta}{2}\right)}$$

$$= \frac{\cos\frac{\theta}{2} - j\sin\frac{\theta}{2}}{-2j\sin\frac{\theta}{2}}$$

$$= \; = -\frac{1}{2j}\cot\frac{\theta}{2} + \frac{1}{2}$$

$$= \frac{1}{2}\left(1 + j\cot\frac{\theta}{2}\right)$$

1) ii)

$$z^6 = 8j$$

$$z^6 = 8\left(\cos\frac{\pi}{2} + j\sin\frac{\pi}{2}\right)$$

$$z = 8^{\frac{1}{6}} e^{\left(\frac{2k\pi + \frac{\pi}{2}}{6}\right)} = \sqrt{2}\, e^{\frac{4k\pi + \pi}{12} j}$$

$z_1 = \sqrt{2}\, e^{\frac{\pi}{12} j}$

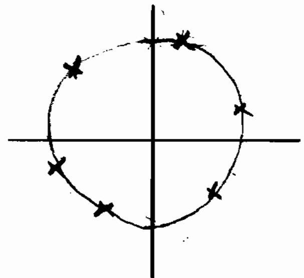
$z_2 = \sqrt{2}\, e^{\frac{5\pi}{12} j}$

$z_3 = \sqrt{2}\, e^{\frac{9\pi}{12} j}$

$z_4 = \sqrt{2}\, e^{\frac{13\pi}{12} j} = \sqrt{2}\, e^{-\frac{11\pi}{12} j}$

$z_5 = \sqrt{2}\, e^{\frac{17\pi}{12} j} = \sqrt{2}\, e^{-\frac{7\pi}{12} j}$

$z_6 = \sqrt{2}\, e^{\frac{21\pi}{12} j} = \sqrt{2}\, e^{-\frac{3\pi}{12} j}$



Regular hexagon on circle radius $\sqrt{2}$

1) iii) $\sqrt{2}\, e^{\frac{9\pi}{12} j} = \sqrt{2}\, e^{\frac{3\pi}{4} j}$

$$= \sqrt{2}\left(\cos\frac{3\pi}{4} + j\sin\frac{3\pi}{4}\right)$$

$$= \sqrt{2}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} j\right)$$

$$= -1 + j$$

Also $\sqrt{2}\, e^{-\frac{3\pi}{12} j} = \sqrt{2}\, e^{-\frac{\pi}{4} j}$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} - j\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} j\right)$$

$$= 1 - j$$

1) iv) Let $z = \left(\sqrt{2} - \frac{1}{w}\right)$

$\Rightarrow \quad wz = \sqrt{2}w - 1$

$$1 = \sqrt{2}w - wz$$

$$1 = w(\sqrt{2} - z)$$

$$w = \frac{1}{\sqrt{2} - z}$$

$$w = \frac{1}{\sqrt{2} - \sqrt{2}e^{\alpha j}}$$

where $\alpha = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{9\pi}{12}, -\frac{11\pi}{12}, -\frac{7\pi}{12}, -\frac{3\pi}{12}$

11 iv cont

$$w = \frac{1}{\sqrt{2}}\left(\frac{1}{1-e^{\alpha j}}\right)$$

$$w = \frac{1}{\sqrt{2}} \times \frac{1}{2}\left(1 + \cot\frac{\alpha}{2}\right)$$

$$w = \frac{1}{2\sqrt{2}}\left(1 + \cot\frac{\alpha}{2}\right)$$

$$\frac{\alpha}{2} = \frac{\pi}{24}, \frac{5\pi}{24}, \frac{9\pi}{24},$$

$$-\frac{11\pi}{24}, -\frac{7\pi}{24}, -\frac{3\pi}{24}$$