1 i) $\ln x$ and its derivatives are not defined for $x = 0$

ii) a) Let $f(x) = \ln(1+x)$ then $f(0) = 0$

$f'(x) = \frac{1}{1+x}$ $\quad f'(0) = 1$

$f''(x) = -\frac{1}{(1+x)^2}$ $\quad f''(0) = -1$

$f'''(x) = +\frac{2}{(1+x)^3}$ $\quad f^{(3)}(0) = 2$

$f^{(4)}(x) = -\frac{2\times 3}{(1+x)^4}$ $\quad f^{(4)}(0) = 6$

Maclaurin series is given by

$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \ldots$

$= 0 + x - \frac{x^2}{2!} + \frac{2x^3}{3!} - \frac{6x^4}{4!} + \ldots$

$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots + \frac{(-1)^{n+1}x^n}{n} + \ldots$

Let $f(x) = \frac{e^x}{e^x+1}$

$f'(x) = \frac{(e^x+1)e^x - e^x e^x}{(e^x+1)^2}$

$= \frac{e^{2x} + e^x - e^{2x}}{(e^x+1)^2}$

$= \frac{e^x}{(e^x+1)^2}$

$f''(x) = \frac{(e^x+1)^2 e^x - e^x 2(e^x+1)e^x}{(e^x+1)^4}$

$= \frac{(e^x+1)e^x - 2e^{2x}}{(e^x+1)^3}$

$= \frac{e^{2x} + e^x - 2e^{2x}}{(e^x+1)^3}$

$= \frac{e^x - e^{2x}}{(e^x+1)^3}$

$f'''(x)$

$= \frac{(e^x+1)^3(e^x - 2e^{2x}) - (e^x - e^{2x})3(e^x+1)^2 e^x}{(e^x+1)^6}$

$= \frac{(e^x+1)(e^x - 2e^{2x}) - 3e^x(e^x - e^{2x})}{(e^x+1)^4}$

$= \frac{e^{2x} + e^x - 2e^{3x} - 2e^{2x} - 3e^{2x} + 3e^{3x}}{(e^x+1)^4}$

$= \frac{e^x - 4e^{2x} + e^{3x}}{(e^x+1)^4}$

$\Rightarrow f(0) = \frac{1}{2}, \quad f'(0) = \frac{1}{4}$

$f''(0) = 0 \qquad f'''(0) = -\frac{2}{2^4} = -\frac{1}{8}$

Series is given by

$f(x) = \frac{1}{2} + \frac{x}{4} - \frac{x^3}{8\times 3!} + \ldots$

$= \frac{1}{2} + \frac{x}{4} - \frac{x^3}{48} + \ldots$

3 cont) Show $f(x) - f(0)$ is an odd function

Let $F(x) = f(x) - f(0)$

$$f(x) - f(0) = \frac{e^x}{e^x+1} - \frac{1}{2}$$

$$= \frac{2e^x - (e^x+1)}{2(e^x+1)}$$

$$F(x) = \frac{e^x - 1}{2(e^x+1)}$$

$$\therefore F(-x) = \frac{e^{-x} - 1}{2(e^{-x}+1)}$$

Multiply by $\frac{e^x}{e^x}$

$$F(-x) = \frac{1 - e^x}{2(1+e^x)} = -F(x)$$

$\therefore F(x)$ is an odd function and cannot contain even powers of x

$f(x) - f(0)$ is $\therefore$ an odd fn

$\therefore f(x) = f(0) +$ an odd fn

$\therefore$ no even powers of x

4)

i) $\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - + \ldots - \frac{x^{10}}{10}$

When $x = 1$

$\ln 2 \approx 0.6456$

ii) $\ln 2 = -\ln\left(\frac{1}{2}\right) = -\ln\left(1 - \frac{1}{2}\right)$

$$\ln 2 \approx -\left[\left(-\tfrac{1}{2}\right) - \frac{\left(-\frac{1}{2}\right)^2}{2} + \frac{\left(-\frac{1}{2}\right)^3}{3} - \frac{\left(-\frac{1}{2}\right)^4}{4} + \frac{\left(-\frac{1}{2}\right)^5}{5} - \frac{\left(-\frac{1}{2}\right)^6}{6}\right]$$

$$\ln 2 \approx -\left[-\frac{1}{2} - \frac{1}{8} - \frac{1}{24} - \frac{1}{64} - \frac{1}{160} - \frac{1}{384}\right]$$

$\ln 2 \approx 0.6911$

4 iii)

$$\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6}$$

$$\ln(1-x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6}$$

$$\ln(1+x) - \ln(1-x)$$

$$\approx 2x + \frac{2x^3}{3} + \frac{2x^5}{5}$$

We need $\ln\left(\frac{1+x}{1-x}\right) = \ln 2$

$$\Rightarrow \quad 2 = \frac{1+x}{1-x}$$

$$2(1-x) = 1+x$$
$$2 - 2x = 1 + x$$
$$1 = 3x$$
$$\tfrac{1}{3} = x$$

$$\therefore \ln 2 = \ln\left(1+\tfrac{1}{3}\right) - \ln\left(1 - \tfrac{1}{3}\right)$$

$$\approx 2 \times \frac{1}{3} + \frac{2\left(\frac{1}{3}\right)^3}{3} + \frac{2\left(\frac{1}{3}\right)^5}{5}$$

$$= 0.6930$$

5) i) $f(x) = \arctan(1+x)$

$$f'(x) = \frac{1}{1+(x+1)^2}$$

$$f'(x) = \frac{1}{x^2+2x+2}$$

$$f''(x) = \frac{-(2x+2)}{(x^2+2x+2)^2}$$

5 ii) $\Rightarrow f(0) = \arctan 1 = \frac{\pi}{4}$

$$f'(0) = \frac{1}{2}$$

$$f''(0) = \frac{-2}{4} = -\frac{1}{2}$$

$$\therefore \arctan x \approx \frac{\pi}{4} + \frac{x}{2} - \frac{x^2}{4}$$

5 iii)

$$\int_0^{0.4} \arctan(1+x^2)\,dx$$

$$\approx \int_0^{0.4} \left(\frac{\pi}{4} + \frac{x^2}{2} - \frac{x^4}{4}\right) dx$$

$$= \left[\frac{\pi}{4}x + \frac{x^3}{6} - \frac{x^5}{20}\right]_0^{0.4}$$

$$= 0.1\pi + \frac{0.4^3}{6} - \frac{0.4^5}{20}$$

$$= 0.324$$

6 $\frac{dy}{dx} = 1 - xy$ ①

Passes through $(0, 2)$

Assume $y = a_0 + a_1 x + a_2 x^2 + +$

i) Since curve passes through $(0,2)$

$$\Rightarrow a_0 = 2$$

From series

$$\frac{dy}{dx} = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 +$$

Subst in ①

$$\frac{dy}{dx} = 1 - x\left(a_0 + a_1 x_1 + a_2 x^2 + \dots\right)$$

$$\frac{dy}{dx} = 1 - a_0 x - a_1 x^2 - a_2 x^3 - \dots$$

$$= 1 - 2x - a_1 x^2 - a_2 x^3 - \dots$$

$$\therefore (a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \dots)$$
$$\equiv (1 - 2x - a_1 x^2 - a_2 x^3 - \dots)$$

ii) $\Rightarrow a_1 = 1, \quad a_2 = -1$

$$3a_3 = -1 \Rightarrow a_3 = -\frac{1}{3}$$

$$4a_4 = 1 \Rightarrow a_4 = \frac{1}{4}$$

$$5a_5 = \frac{1}{3} \Rightarrow a_5 = \frac{1}{15}$$

$$6a_6 = -\frac{1}{4} \Rightarrow a_6 = -\frac{1}{24}$$

First 7 terms

$$y = 2 + x - x^2 - \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{15} - \frac{x^6}{24}$$

6iii) Graphs omitted.

7)

a) $\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + -$

b) $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + -$

c) $\cos\theta + j\sin\theta$

$= 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!} + j\frac{\theta^7}{7!}$

7ii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + +$

$e^{j\theta} = 1 + j\theta - \frac{\theta^2}{2!} - j\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + j\frac{\theta^5}{5!} - \frac{\theta^6}{6!}$

iii) Coeffs match for all powers of θ

$\therefore e^{j\theta} = \cos\theta + j\sin\theta$

8)

i) $f(x) = \arcsin x$

$f'(x) = \frac{1}{\sqrt{1-x^2}}$

$\therefore y_1 = \frac{1}{\sqrt{1-x^2}}$

$\Rightarrow y_1^2 = \frac{1}{1-x^2}$

$\Rightarrow y_1^2(1-x^2) = 1$

Since $y_1 = (1-x^2)^{-\frac{1}{2}}$

$y_2 = -\frac{1}{2}(1-x^2)^{-3/2}(-2x)$

$y_2 = x(1-x^2)^{-3/2}$

$(1-x^2)y_2 = x(1-x^2)^{-\frac{1}{2}}$

$(1-x^2)y_2 = xy_1$

$\therefore (1-x^2)y_2 - xy_1 = 0$

8ii)

$a_1 = f'(0) = 1$

$a_2 = f''(0) = 0$

8iii) Assume

$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2y_n = 0$ for some integer n

Differentiate

$(1-x^2)y_{n+3} + y_{n+2}(-2x)$

$-(2n+1)x\,y_{n+2} - (2n+1)y_{n+1}$

$-n^2y_{n+1} = 0$

$\Rightarrow (1-x^2)y_{n+3} - (2n+2+1)y_{n+2}$

$-(2n+1+n^2)y_{n+1} = 0$

$\Rightarrow (1-x^2)y_{n+1+2} - (2(n+1)+1)y_{n+1+1}$

$-(n+1)^2y_{n+1} = 0$

Same form with $n+1$ replacing n

$\therefore$ if true for n then true for $n+1$

When $n=0$

$(1-x^2)y^2 - xy_1 - 0^2y_n = 0$ from part (i)

8 iii) cont) $\therefore$ by induction true for all positive integers n

Putting $x=0$

$$1y_{n+2} - (2n+1)0y_{n+1} - n^2y_n = 0$$

$$a_{n+2} - 0 - n^2 a_n = 0$$

$$\therefore \quad a_{n+2} = n^2 a_n$$

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8 iv)

$$a_0 = \arcsin 0 = 0$$

$$a_1 = f'(0) = 1$$

$$a_2 = f''(0) = 0$$

Now $a_3 = 1^2 a_1 = 1$

$$a_4 = 2^2 a_2 = 0$$

$$a_5 = 3^2 a_3 = 9$$

First 3 non-zero terms

$$\arcsin x \approx x + \frac{x^3}{3!} + \frac{9x^5}{5!}$$

General term omitted.