1 i)

$$\underline{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 4 & 5 \\ 0 & 1 & 2 \end{pmatrix}$$

$A_1 = 8-5 = 3$
$A_2 = -(4-4) = 0$
$A_3 = 10-16 = -6$

$\text{Det } \underline{M} = 1\times 3 + 2\times 0 + 0\times -6$
$= 3$

$B_1 = -(4-0) = -4$
$B_2 = 2-0 = 2$
$B_3 = -(5-8) = 3$

$C_1 = 2-0 = 2$
$C_2 = -(1-0) = -1$
$C_3 = 4-4 = 0$

$$\text{adj}\,\underline{M} = \begin{pmatrix} 3 & 0 & -6 \\ -4 & 2 & 3 \\ 2 & -1 & 0 \end{pmatrix}$$

$$\underline{M}^{-1} = \begin{pmatrix} 1 & 0 & -2 \\ -\frac{4}{3} & 2/3 & 1 \\ \frac{2}{3} & -\frac{1}{3} & 0 \end{pmatrix}$$

---

1 ii)

$$\underline{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 6 \\ 5 & 3 & 11 \\ 7 & 4 & 16 \end{pmatrix}$$

$A_1 = 48-44 = 4$
$A_2 = -(32-24) = -8$
$A_3 = 22-18 = 4$

$\det \underline{M} = 3\times 4 + 5\times(-8) + 7\times 4$
$= 12 - 40 + 28 = 0$
$\therefore\ \underline{M}$ is singular

---

1 iii)

$$\underline{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 5 & 5 & -5 \\ -9 & 3 & -5 \\ -4 & -6 & 8 \end{pmatrix}$$

$A_1 = 24-30 = -6$
$A_2 = -(40-30) = -10$
$A_3 = -25 - -15 = -10$

$\det \underline{M} = 5\times(-6) + (-9)\times(-10) + (-4)\times(-10)$
$= -30 + 90 + 40 = 100$

$B_1 = -(-72-20) = 92$
$B_2 = 40-20 = 20$
$B_3 = -(-25-45) = 70$

$C_1 = 54 - -12 = 66$
$C_2 = -(-30 - -20) = 10$
$C_3 = 15 - -45 = 60$

$$\underline{M}^{-1} = \frac{1}{100}\begin{pmatrix} -6 & -10 & -10 \\ 92 & 20 & 70 \\ 66 & 10 & 60 \end{pmatrix}$$

$$= \begin{pmatrix} -0.06 & -0.1 & -0.1 \\ 0.92 & 0.2 & 0.7 \\ 0.66 & 0.1 & 0.6 \end{pmatrix}$$

---

1 iv)

$$\underline{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 6 & 5 & 6 \\ -5 & 2 & -4 \\ -4 & -6 & -5 \end{pmatrix}$$

$A_1 = -10-24 = -34$
$A_2 = -(-25 - -36) = -11$
$A_3 = -20-12 = -32$

$\det \underline{M} = 6\times(-34) - 5\times(-11) - 4\times(-32)$
$= -204 + 55 + 128 = -21$

1iv cont)
$B_1 = -(25-16) = -9$
$B_2 = -30 - -24 = -6$
$B_3 = -(-24 - -30) = -6$

$C_1 = 30 - -8 = 38$
$C_2 = -(-36 - -20) = 16$
$C_3 = 12 - -25 = 37$

Check
$\det \underline{M} = 5 \times (-9) + 2 \times (-6) + (-6) \times (-6)$

$= -45 - 12 + 36 = -21$

$\det \underline{M} = 6 \times (38) - 4 \times (16) - 5 \times 37$
$= 228 - 64 - 185$
$= -21$

$$\underline{M}^{-1} = -\frac{1}{21}\begin{pmatrix} -34 & -11 & -32 \\ -9 & -6 & -6 \\ 38 & 16 & 37 \end{pmatrix}$$

$$= \frac{1}{21}\begin{pmatrix} 34 & 11 & 32 \\ 9 & 6 & 6 \\ -38 & -16 & -37 \end{pmatrix}$$

2)
$$\underline{M} = \begin{pmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 4 & 1 & 1 \end{pmatrix}$$

$A_1 = 2-3 = -1$
$A_2 = -(2-2) = 0$
$A_3 = 6-4 = 2$

$B_1 = -(3-12) = 9$
$B_2 = 1-8 = -7$
$B_3 = -(3-6) = 3$

$C_1 = 3-8 = -5$
$C_2 = -(1-8) = 7$
$C_3 = 2-6 = -4$

$$\text{adj}\,\underline{M} = \begin{pmatrix} -1 & 0 & 2 \\ 9 & -7 & 3 \\ -5 & 7 & -4 \end{pmatrix}$$

ii)
$\underline{M}\,\text{adj}\,\underline{M}$

$$= \begin{pmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 4 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 & 2 \\ 9 & -7 & 3 \\ -5 & 7 & -4 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

iii) $\text{adj}\,\underline{M}\,\underline{M}$

$$= \begin{pmatrix} -1 & 0 & 2 \\ 9 & -7 & 3 \\ -5 & 7 & -4 \end{pmatrix}\begin{pmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 4 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

iv) $\det \underline{M} = 1 \times (-1) + 3 \times 0 + 4 \times 2$
$= -1 + 0 + 8 = 7$

v)
$\det(\text{adj}\,\underline{M})$

$= -1 \times (28 - 21) - 9 \times (0 - 14) - 5(0 - -14)$

$= -7 + 126 - 70 = 49$

2 vi) Find adj(adj $\underline{M}$)

$$\text{adj}\,\underline{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 9 & -7 & 3 \\ -5 & 7 & -4 \end{pmatrix}$$

$A_1 = 28 - 21 = 7$
$A_2 = -(0 - 14) = 14$
$A_3 = 0 - -14 = 14$

$B_1 = -(-36 - -15) = 21$
$B_2 = 4 - -10 = 14$
$B_3 = -(-3 - 18) = 21$

$C_1 = 63 - 35 = 28$
$C_2 = -(-7 - 0) = 7$
$C_3 = 7 - 0 = 7$

$$\text{adj}(\text{adj}\,\underline{M}) = \begin{pmatrix} 7 & 14 & 14 \\ 21 & 14 & 21 \\ 28 & 7 & 7 \end{pmatrix}$$

$$= 7\begin{pmatrix} 1 & 2 & 2 \\ 3 & 2 & 3 \\ 4 & 1 & 1 \end{pmatrix}$$

$$= (\det \underline{M})\,\underline{M}$$

---

3) Let $\underline{M} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} 4 & -5 & 3 \\ 3 & 3 & -4 \\ 5 & 4 & -6 \end{pmatrix}$

$A_1 = -18 - -16 = -2$
$A_2 = -(30 - 12) = -18$
$A_3 = 20 - 9 = 11$

$\det \underline{M} = 4 \times (-2) + 3 \times (-18) + 5 \times 11$
$= -8 - 54 + 55$
$= -7$

$B_1 = -(-18 - -20) = -2$
$B_2 = -24 - 15 = -39$
$B_3 = -(-16 - 9) = 25$

Check
$\det \underline{M} = -5 \times (-2) + 3(-39) + 4(25)$
$= 10 - 117 + 100 = -7$

$C_1 = 12 - 15 = -3$
$C_2 = -(16 - -25) = -41$
$C_3 = 12 - -15 = 27$

Check
$\det \underline{M} = 3 \times (-3) - 4(-41) - 6(27)$
$= -9 + 164 - 162 = -7$

$$\text{adj}\,\underline{M} = \begin{pmatrix} -2 & -18 & 11 \\ -2 & -39 & 25 \\ -3 & -41 & 27 \end{pmatrix}$$

$$M^{-1} = \frac{1}{7}\begin{pmatrix} 2 & 18 & -11 \\ 2 & 39 & -25 \\ 3 & 41 & -27 \end{pmatrix}$$

$$\begin{pmatrix} 4 & -5 & 3 \\ 3 & 3 & -4 \\ 5 & 4 & -6 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 48 \\ 74 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 2 & 18 & -11 \\ 2 & 39 & -25 \\ 3 & 41 & -27 \end{pmatrix}\begin{pmatrix} 3 \\ 48 \\ 74 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 56 \\ 28 \\ -21 \end{pmatrix} \Rightarrow \begin{aligned} x &= 8 \\ y &= 4 \\ z &= -3 \end{aligned}$$

4) $\underline{A} = \begin{pmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{pmatrix}$

$\underline{A}\underline{A}^{-1} = \begin{pmatrix} 6 & 2 & -3 \\ 2 & 3 & 6 \\ 3 & -6 & 2 \end{pmatrix}\begin{pmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{pmatrix}$

$= \begin{pmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{pmatrix}$

i) $\underline{A}^{-1} = \frac{1}{49}\begin{pmatrix} 6 & 2 & 3 \\ 2 & 3 & -6 \\ -3 & 6 & 2 \end{pmatrix}$

ii) $\det \underline{A} = 49$

iii) $A^T A = \begin{pmatrix} 49 & 0 & 0 \\ 0 & 49 & 0 \\ 0 & 0 & 49 \end{pmatrix} = 49\,\underline{I}_3$

5) $\underline{P} = \begin{pmatrix} -2 & 1 & 0 \\ 3 & 2 & 5 \\ 1 & 2 & 2 \end{pmatrix}$ $\underline{Q} = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix}$

$\underline{P}\underline{Q} = \begin{pmatrix} -5 & 4 & 6 \\ 14 & 16 & 13 \\ 4 & 9 & 8 \end{pmatrix}$

$A_1 = 16\times 8 - 9\times 13 = 11$
$A_2 = -(32 - 54) = 22$
$A_3 = 52 - 96 = -44$

$\det(\underline{P}\underline{Q}) = -5(11) + 14(22) + 4(-44)$

$= -55 + 308 - 176 = 77$

$\underline{Q}\underline{P} = \begin{pmatrix} -9 & -4 & -9 \\ 10 & 7 & 14 \\ 8 & 14 & 21 \end{pmatrix}$

$A_1 = 7\times 21 - 14\times 14 = -49$
$A_2 = -(-4\times 21 - -9\times 14) = -42$
$A_3 = -4\times 14 - -9\times 7 = 7$

$\det(\underline{Q}\underline{P}) = -9\times(-49) + 10(-42) + 8(7)$

$= 441 - 420 + 56 = 77$

$\underline{P} = \begin{pmatrix} -2 & 1 & 0 \\ 3 & 2 & 5 \\ 1 & 2 & 2 \end{pmatrix}$

$A_1 = 4 - 10 = -6$
$A_2 = -(2 - 0) = -2$
$A_3 = 5 - 0 = 5$

$\det \underline{P} = -2(-6) + 3(-2) + 1(5)$
$= 12 - 6 + 5 = 11$

$\underline{Q} = \begin{pmatrix} 2 & -1 & -2 \\ -1 & 2 & 2 \\ 2 & 3 & 3 \end{pmatrix}$

$A_1 = 6 - 6 = 0$
$A_2 = -(-3 - -6) = -3$
$A_3 = -2 - -4 = 2$

$\det \underline{Q} = 2(0) - 1(-3) + 2(2)$
$= 0 + 3 + 4 = 7$

$\therefore \det \underline{P} \times \det \underline{Q} = 11\times 7 = 77$
$= \det(\underline{P}\underline{Q}) = \det(\underline{Q}\underline{P})$

6) i
$$\underline{A} = \begin{pmatrix} 5 & 0 & 3 \\ 2 & x & 0 \\ 3 & 4 & 5 \end{pmatrix}$$

Expand down column B

$$\det \underline{A} = -0\times 10 + x(25-9) - 4(0-6)$$

$$\det \underline{A} = 0 + 16x + 24$$

Singular so $\det \underline{A} = 0$

$$\Rightarrow 16x = -24$$

$$x = -\frac{3}{2}$$

6 ii)
$$\underline{A} = \begin{pmatrix} 4 & 6 & -1 \\ -1 & 2 & -3 \\ 5 & x & 15 \end{pmatrix}$$

Expand down column B

$$\det \underline{A} = 0 = -6(-15 - -15) + 2(60 - -5) - x(-12-1)$$

$$0 = 0 + 130 + 13x$$

$$\Rightarrow 13x = -130$$

$$x = -10$$

6 iii)
$$\underline{A} = \begin{pmatrix} 6 & 7 & -1 \\ 3 & x & 5 \\ 9 & 11 & x \end{pmatrix}$$

Expand down column A

$$\det \underline{A} = 0 = 6(x^2-55) - 3(7x+11) + 9(35+x)$$

$$0 = 2(x^2-55) - (7x+11) + 3(35+x)$$

$$2x^2 - 110 - 7x - 11 + 105 + 3x = 0$$

$$2x^2 - 4x - 16 = 0$$

$$x^2 - 2x - 8 = 0$$

$$(x-4)(x+2) = 0$$

$$\Rightarrow x = 4 \text{ or } x = -2$$

6 iv)
$$\underline{A} = \begin{pmatrix} 1-x & 1 & -2 \\ -1 & 2-x & 1 \\ 0 & 1 & -1-x \end{pmatrix}$$

Expand down column A

$$\det \underline{A} = 0 = (1-x)\big((2-x)(-1-x) - 1\big) + 1\big(1(-1-x) + 2\big) + 0$$

$$0 = (1-x)(-2 + x - 2x + x^2 - 1) + (-1 - x + 2)$$

$$0 = (1-x)(x^2 - x - 3) + (1-x)$$

$$0 = (1-x)(x^2 - x - 3 + 1)$$

$$0 = (1-x)(x^2 - x - 2)$$

$$0 = (1-x)(x-2)(x+1)$$

$$\Rightarrow x = 1 \text{ or } x = 2 \text{ or } x = -1$$

7) i) If $\underline{M}$ non-singular then $\det \underline{M} \neq 0$

If $\underline{N}$ non-singular then $\det \underline{N} \neq 0$

$\therefore \det \underline{N} \times \det \underline{M} = \det \underline{M}\underline{N} \neq 0$

$\therefore (\underline{M}\underline{N})$ is non-singular

7 ii)

a) $(\underline{M}\underline{N})(\underline{N}^{-1}\underline{M}^{-1})$

$= \underline{M}\,\underline{I}\,\underline{M}^{-1} = \underline{M}\underline{M}^{-1} = \underline{I}$

$\therefore (\underline{M}\underline{N})^{-1} = (\underline{N}^{-1}\underline{M}^{-1})$

b) $(\underline{M}\underline{N})^{-1} = \dfrac{1}{\det(\underline{M}\underline{N})} \operatorname{adj}(\underline{M}\underline{N})$

$\underline{N}^{-1} = \dfrac{1}{\det \underline{N}} \operatorname{adj} \underline{N}$

$\underline{M}^{-1} = \dfrac{1}{\det \underline{M}} \operatorname{adj} \underline{M}$

From part a)

$\dfrac{1}{\det(\underline{M}\underline{N})} \operatorname{adj}(\underline{M}\underline{N}) = \dfrac{1}{\det \underline{N}} \operatorname{adj}(\underline{N}) \dfrac{1}{\det \underline{M}} \operatorname{adj}(\underline{M})$

$\therefore \dfrac{1}{\det(\underline{M}\underline{N})} \operatorname{adj}(\underline{M}\underline{N}) = \dfrac{1}{\det \underline{N}} \cdot \dfrac{1}{\det \underline{M}} \operatorname{adj} \underline{N} \operatorname{adj} \underline{M}$

But $\det(\underline{M}\underline{N}) = \det \underline{N} \times \det \underline{M}$

$\therefore \operatorname{adj}(\underline{M}\underline{N}) = \operatorname{adj} \underline{N} \operatorname{adj} \underline{M}$

8) Given $\underline{M}\underline{M}^T = \underline{M}^T\underline{M}$

i) $\underline{M}\underline{M}^T\underline{M}^{-1} = \underline{M}^T\underline{M}\underline{M}^{-1}$

$\Rightarrow \underline{M}\underline{M}^T\underline{M}^{-1} = \underline{M}^T\underline{I} = \underline{M}^T$

$\Rightarrow \underline{M}^{-1}\underline{M}\underline{M}^T\underline{M}^{-1} = \underline{M}^{-1}\underline{M}^T$

$\Rightarrow \underline{I}\,\underline{M}^T\underline{M}^{-1} = \underline{M}^{-1}\underline{M}^T$

$\Rightarrow \underline{M}^T\underline{M}^{-1} = \underline{M}^{-1}\underline{M}^T$

8 ii) If $\underline{N} = \underline{M}^{-1}\underline{M}^T$

then $\underline{N}\underline{N}^T = (\underline{M}^{-1}\underline{M}^T)(\underline{M}^{-1}\underline{M}^T)^T$

$= (\underline{M}^{-1}\underline{M}^T)((\underline{M}^T)^T(\underline{M}^{-1})^T)$

$= (\underline{M}^{-1}\underline{M}^T)(\underline{M}(\underline{M}^T)^{-1})$

$= \underline{M}^{-1}\underline{M}\,\underline{M}^T(\underline{M}^T)^{-1}$

$= \underline{I} \times \underline{I} = \underline{I}$

9) $\underline{P} = \begin{pmatrix} a & 0 & b \\ 0 & c & 0 \\ d & 0 & e \end{pmatrix}$ $\underline{Q} = \begin{pmatrix} e & 0 & -b \\ 0 & f & 0 \\ -d & 0 & a \end{pmatrix}$

i) $\underline{P}\underline{Q} = \begin{pmatrix} ae-bd & 0 & 0 \\ 0 & cf & 0 \\ 0 & 0 & ae-bd \end{pmatrix}$

ii) $cf = ae - bd$

$f = \dfrac{ae-bd}{c}$

$c \neq 0$, $ae \neq bd$

9 iii) $\underline{P}^{-1} =$

$$\frac{1}{ae-bd}\begin{pmatrix} e & 0 & -b \\ 0 & \frac{ae-bd}{c} & 0 \\ -d & 0 & a \end{pmatrix}$$

9 iv)

$$\underline{M} = \begin{pmatrix} 3 & 0 & 8 \\ 0 & 2 & 0 \\ 1 & 0 & 4 \end{pmatrix} \quad N^{-1} = \begin{pmatrix} 1 & -2 & 3 \\ 3 & k & 0 \\ 2 & 4 & k \end{pmatrix}$$

From part iii

$$\underline{M}^{-1} = \frac{1}{12-8}\begin{pmatrix} 4 & 0 & -8 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$$M^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 0 & -8 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$$(MN)^{-1} = N^{-1}M^{-1}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & -2 & 3 \\ 3 & k & 0 \\ 2 & 4 & k \end{pmatrix}\begin{pmatrix} 4 & 0 & -8 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

$$= \frac{1}{4}\begin{pmatrix} 1 & -4 & 1 \\ 12 & 2k & -24 \\ 8-k & 8 & -16+3k \end{pmatrix}$$

9 v)

$$\underline{MN}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 2k \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & -4 & 1 \\ 12 & 2k & -24 \\ 8-k & 8 & -16+3k \end{pmatrix}\begin{pmatrix} 0 \\ 2k \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{4}\begin{pmatrix} -8k \\ 4k^2 \\ 16k \end{pmatrix}$$

$x = -2k$

$y = k^2$

$z = 4k$